**5.3**

**9. Proof (by mathematical induction):**

For the given statement, the property is “7n – 1 is divisible by 6.”

**Basic Step:**

P(0) = 70 – 1 = 1 – 1 = 0

0 is divisible by 6 because 0 = 6 ∙ 0

Thus P(0) is true.

**Induction Step:**

Let k be any integer with k ≥ 0, and suppose P(k) is true. That is, suppose 7k – 1 is divisible by 6.

P(k + 1) = 7k+1 – 1

= 7k ∙ 7 – 1

= 7k ∙ (6 + 1) – 1

= 7k ∙ 6 + 7k – 1 (equation 9.1)

By the inductive hypothesis 7k – 1 is divisible by 6, and so 7k – 1 = 6r for some integer r. By substitution into equation 9.1:

7k+1 – 1 = 7k ∙ 6 + 6r = 6 ∙ (7k + r)

Since k and r are integers (7k + r) is an integer. Hence, by definition of divisibility 7k+1 – 1 is divisible by 6.

**5.6**

**20a.** Using k = 2 and since C1 = 1 and C2 = 5, the minimum number of steps required to move two disks,

C2 ≤ 4 ∙ C1 + 1

5 ≤ 4 ∙ 1 + 1

5 ≤ 5

With C3 = 15, the minimum number of steps required to move three disks,

C3 ≤4 ∙ C2 + 1

15 ≤ 4 ∙ 5 + 1

15 ≤ 21

So, for all integers k ≥ 2, the inequality Ck ≤ 4 ∙ Ck-1 + 1 holds.

**5.7**

**13.**

Tk = Tk-1 + 3k + 1, for all k ≥ 1 and T0 = 0

Tk-1 = Tk-2 + 3(k – 1) + 1 (equation 13.1)

Tk-2 = Tk-3 + 3(k – 2) + 1 (equation 13.2)

Tk-3 = Tk-4 + 3(k – 3) + 1 (equation 13.3)

Tk = Tk-1 + 3k + 1

= Tk-2 + 3(k – 1) + 1 + 3k + 1 (substituting 13.1 for Tk-1)

= Tk-2 + 3[(k – 1) + k] + 1 ∙ 2

= Tk-3 + 3(k – 2) + 1 + 3[(k – 1) + k] + 1 ∙ 2 (substituting 13.2 for Tk-2)

= Tk-3 + 3[(k – 2)(k – 1) + k] + 1 ∙ 3

Tk-k + 3[(k(k+1))/2] + 1 ∙ k

= 0 + 3[(k(k+1))/2] + 1 ∙ k (since Tk-k = T0 = 0)

= 3[(k(k+1))/2] + k

**5.8**

**12.** This is a distinct roots case, using Theorem 5.8.3, t2 – 0t – 9 = 0 is the characteristic equation. Factoring leads to (x – 3)(x + 3) = 0 so, r = 3 and s = -3.

En = C ∙ 3n + D ∙ (-3)n

E0 = 0 = C + D so, C = -D

E1 = 2 = C ∙ 31 + D ∙ (-3)1

2 = -3D – 3D (substituting C = -D)

D = -1/3

Therefore, C = 1/3 and the formula is Ek = (1/3) ∙ 3k + (-1/3) ∙ (-3)k for all integers k ≥ 2

**15.** This is a single roots case, using Theorem 5.8.5, t2 – 6t + 9 = 0 is the characteristic equation.

Factoring leads to (t – 3)2 = 0 so, r = s = 0.

Tn = C ∙ 3n + D ∙ n ∙ 3n

T0 = 1 = C ∙ 30 + D ∙ 0 ∙ 30 so, C = 1

T1 = 3 = C ∙ 31 + D ∙ 1 ∙ 31

3 = 3C + 3D

0 = 3D (substituting C = 1 and subtracting from LHS)

D = 0

Therefore, the formula is Tk = 1 ∙ 3k + 0 ∙ k ∙ 3k = 3k for all integers k ≥ 2

**Implementation Problem**

|  |  |
| --- | --- |
| Processor Speed | 2.5 GHz |
| No. of cores | 8 |
| Sze of Memory | 16 GB |
| Operating System | Windows 8.1 |
| Programming Language | Java |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| N | Fn | Elapsed Time with Recursive Method (ms) | Elapsed Time with Inductive Method (ms) | Elapsed Time with Formula (ms) |
| 10 | 55 | 0.008 | 0.0032 | 0.0148 |
| 20 | 6765 | 0.42 | 0.0049 | 0.0189 |
| 30 | 832040 | 4.94 | 0.0053 | 0.0229 |
| 40 | 102334155 | 492.7 | 0.0061 | 0.025 |
| 50 | 12586269025 | 59251.73 | 0.0065 | 0.0201 |

I preferred writing the code for the inductive method, probably because it was just a simple for loop. The recursive method was also rather easy to code, while the formula was messy mainly because of calling the pow and sqrt methods from the Math library. The elapsed times for each method varied a bit, it would probably have been better to take an average. Presumably the formula should be a constant time.